

## ON A QUESTION OF KOLLÁR.

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ABSTRACT. We show: If a bounded domain in a Stein space covers a compact complex space, it must be smooth. This give a negative answer to a question of Kollár.

## 1. INTRODUCTION

The theory of the Kobayashi pseudodistance provides a link between complex analysis and metric topology. We use this link to discuss two topics.

The Shafarevich conjecture (see [6]) postulates that the universal covering of a projective complex manifold ought to be holomorphically convex. One important piece of evidence for this conjecture is the the following result which was proved by Siegel [7] in 1949: *If the universal covering of a complex compact manifold can be realized as a bounded domain in  $\mathbb{C}^n$ , then this domain is holomorphically convex.*

This result was later improved by Ivashkovich ([3]) who showed: *If the universal covering of a compact Kähler manifold can be realized as an open subset  $D$  in a complex manifold  $M$ , then  $D$  is locally Stein, i.e., for each point  $p \in M$  there exists an open neighbourhood  $W$  of  $p$  in  $M$  such that  $W \cap D$  is Stein or empty.*

Minimal model theory for projective manifolds suggests that one ought to accept “mild” singularities. Hence it is natural to ask, whether this result can be generalized to singular spaces, and moreover to ask, as Kollár did in [4], whether there exist compact spaces with a bounded singular domain as universal covering.

We will prove that this is impossible and we will also provide an alternative proof for Siegel’s result.

Both statements will arise as corollaries of two more technical results.

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1991 *Mathematics Subject Classification.* 32E05, 32J18.

*Acknowledgement.* The author was supported by the Mittag-Leffler-Institute and the DFG Forschergruppe 790 “Classification of algebraic surfaces and compact complex manifolds”.

## 2. THE RESULTS

We present the two more technical results, proposition 1 and 2, from which we deduce the main results.

**Definition.** *The action of a group  $G$  on a topological space  $X$  is said to be “cocompact”, if there exists a compact subset  $K \subset X$  such that  $G \cdot K = X$ .*

**Examples.** (1) *If a group  $G$  acts transitively on a space  $X$ , then the action is cocompact, because one point is compact.*  
 (2) *If  $M$  is a compact (real or complex) manifold, then the action of the fundamental group  $\pi_1(M)$  on the universal covering  $\tilde{M}$  by deck transformations is cocompact.*  
 (3) *Let  $\Gamma$  be a discrete group acting properly discontinuously on a locally compact topological space  $X$  with a compact quotient  $X/\Gamma$  and let  $G$  be a group of homeomorphisms of  $X$  containing  $\Gamma$ . Then the  $G$ -action (as well as the  $\Gamma$ -action) on  $X$  is cocompact.*

**Proposition 1.** *Let  $G$  be a group acting cocompactly by isometries on a locally compact metric topological space  $X$ .*

*Then  $X$  is a complete metric space and for every isometric embedding  $i : X \hookrightarrow Y$  of  $X$  into a metric topological space  $Y$  the image  $i(X)$  is closed in  $Y$ .*

*Proof.* Let  $K$  be a compact subset of  $X$  with  $G \cdot K = X$ .

We define a function  $\rho : X \rightarrow \mathbb{R}$  as follows: For every  $x \in X$  the value  $\rho(x)$  is defined as the supremum of all  $r > 0$  for which the closed ball  $\bar{B}_r(x) = \{p \in X : d(p, x) \leq r\}$  is compact. Note that  $\rho(x) > 0$  for every  $x$  because  $X$  is locally compact. Observe furthermore that

$$\bar{B}_{r-\epsilon}(y) \subset \bar{B}_r(x)$$

if  $d(x, y) \leq \epsilon$ . This easily implies

$$|\rho(x) - \rho(y)| \leq d(x, y) \quad \forall x, y \in X$$

which in turn implies that  $\rho$  is a continuous function. Hence there is a minimum for  $\rho$  on the compact set  $K$ . Since  $\rho$  is evidently invariant under all isometries of  $X$  and  $G \cdot K = X$ , it follows that there exists a constant  $c > 0$  such that  $\rho(x) > c$  for all  $x \in X$ .

Let  $(x_n)$  be a Cauchy sequence in  $X$ . Then there exists a natural number  $N$  such that  $d(x_n, x_m) < c$  if  $n, m \geq N$ . Then  $x_n \in \bar{B}_c(x_N)$  for all  $n \geq N$ . But  $\bar{B}_c(x_N)$  is compact, because  $\rho(x_N) > c$ . Therefore every Cauchy sequence in  $X$  is convergent, i.e.,  $X$  is complete.

To prove the second statement, assume the contrary. Then there exists an isometric embedding  $i : X \rightarrow Y$  and a sequence  $x_n$  in  $i(X)$

which converges to a point in  $Y \setminus i(X)$ . But this would imply that  $(x_n)$  is a Cauchy sequence in  $X$  which does not converge inside  $X$  — a contradiction to the completeness of  $X$ .  $\square$

**Corollary 1.** *A homogeneous locally compact metric space is complete.*

**Corollary 2.** *Let  $X$  be a homogeneous complex manifold.*

*If  $X$  is hyperbolic, the Kobayashi pseudodistance  $d_X$  defines a complete metric on  $X$ .*

*If the bounded holomorphic functions on  $X$  separate the points, the Caratheodory-pseudodistance  $c_X$  defines a complete metric on  $X$ .*

**Corollary 3.** *A homogeneous bounded domain in  $\mathbb{C}^n$  is a complete metric space with respect to both the Kobayashi and the Caratheodory pseudometric.*

**Theorem 1.** *Let  $D$  be a domain (=connected open submanifold) in a hyperbolic Stein manifold  $Z$ . Assume that there is a group  $G$  acting on  $D$  cocompactly by biholomorphic transformations.*

*Then  $D$  is Stein.*

**Remark.** *If  $D$  is a bounded domain in  $\mathbb{C}^n$ , then  $D$  is contained in some ball  $B_R = \{v \in \mathbb{C}^n : \|v\| < R\}$  which is a hyperbolic Stein manifold.*

*Similarly for a bounded domain  $D$  of an arbitrary Stein manifold  $Z$ : There is an embedding  $\zeta : Z \hookrightarrow \mathbb{C}^n$ , and for  $R \gg 0$  the preimage  $\zeta^{-1}(B_R(0)) = \{z \in Z : \|\zeta(z)\| < R\}$  is a hyperbolic Stein manifold containing  $D$  as an open subset.*

*Proof.* Let  $i : D \hookrightarrow E$  be the envelope of holomorphy (see e.g. [2]). There is a natural projection  $\pi : E \rightarrow Z$  which is locally biholomorphic. It follows that  $E$  is also hyperbolic ([5], prop. 3.2.9).

Note that  $E$  is a Stein space with  $\mathcal{O}(E) = \mathcal{O}(D)$  (see e.g. [2]). Hence the points of  $E$  correspond to the closed maximal ideals in the ring of holomorphic functions on  $D$ . Therefore each holomorphic automorphism of  $D$  extends to an automorphism of  $E$ .

Thus  $G$  acts on  $D$  by automorphisms which extend to  $E$  and therefore preserve the Kobayashi distance  $d_E$  on  $E$ . Hence  $D$  equipped with the metric given by  $d_E$  becomes a metric space on which  $G$  acts cocompactly and isometrically. It follows that  $D$  is closed in  $E$ . However,  $D$  is always open in  $E$ , hence  $D$  being closed in  $E$  implies that  $D$  is a union of connected components of  $E$ . Now  $\mathcal{O}(E) \simeq \mathcal{O}(D)$  and  $D$  being connected imply that  $E$  is connected. Thus  $D = E$ . As a consequence,  $D$  is Stein.  $\square$

**Corollary 4** (Siegel [6]). *Let  $M$  be a compact complex manifold with universal covering  $D$ . If  $D$  can be embedded into a Stein manifold  $Z$  as a bounded domain, then  $D$  is Stein.*

**Corollary 5.** *A homogeneous bounded domain in a Stein manifold is itself Stein.*

(This is known due to the fundamental work of Gindikin, Pyatetski-Shapiro and Vinberg on bounded homogeneous domains [1]).

**Proposition 2.** *Let  $D$  be a connected complex space, let  $\Gamma$  be a group of automorphisms of  $D$  acting cocompactly on  $D$  and let  $Z$  denote a non-empty  $\Gamma$ -invariant subset of  $D$ .*

*Then every bounded holomorphic function on  $D$  which vanishes on  $Z$  must be identically zero.*

*Proof.* Fix  $p \in Z$ . Let  $f : D \rightarrow \mathbb{C}$  be a bounded holomorphic function vanishing on  $Z$ . Assume that  $f$  is not constant. Without loss of generality we may assume that  $\sup\{|f(z)| : z \in D\} = 1$ . Let  $\Delta = \{w \in \mathbb{C} : |w| < 1\}$ . Then

$$\sup_{z \in D} d_{\Delta}(f(z), f(p)) = +\infty$$

since

$$d_{\Delta}(w, 0) = \log \frac{1 + |w|}{1 - |w|} \quad (w \in \Delta)$$

(see [5], p.21).

By the definition of the Caratheodory pseudodistance  $c_D$  we have:

$$c_D(q, z) \geq d_{\Delta}(0, f(z))$$

for all  $z \in D, q \in Z$ .

It follows that

$$\rho(z) \stackrel{\text{def}}{=} \inf_{q \in Z} c_D(q, z)$$

is an unbounded continuous function on  $D$ . But  $\rho$  is  $G$ -invariant, because  $Z$  is  $G$ -invariant and

$$c_D(g(z), g(w)) = c_D(z, w) \quad \forall z, w \in D \forall g \in \text{Aut}(D).$$

This leads to a contradiction: Every continuous function is bounded on the compact set  $K$  and therefore  $G \cdot K = D$  implies that every  $G$ -invariant continuous function on  $D$  is bounded.

Hence there exists no such non-constant function  $f$ , i.e., if a bounded holomorphic function  $f$  on  $D$  vanishes along  $Z$ , it must vanish identically on  $D$ .  $\square$

**Corollary 6.** *Let  $D$  be a bounded domain in an irreducible Stein space  $X$ , let  $G$  be a group acting cocompactly on  $D$  and let  $Z$  be a closed analytic subset of  $X$  containing a non-empty  $G$ -invariant subset of  $D$ . Then  $Z = X$ .*

*Proof.* Because  $X$  is Stein,  $Z$  can be defined by global holomorphic functions on  $X$ . The restriction of a holomorphic function on  $X$  to  $D$  is necessarily bounded, because  $D$  is relatively compact in  $X$ .

Thus thm. 2 implies that every holomorphic function vanishing on  $Z$  must also vanish on  $D$ . Hence  $D \subset Z$ , which implies  $Z = X$ , because  $X$  is irreducible.  $\square$

For ball quotients this specializes to the following fact:

**Corollary 7.** *Let  $X$  be a compact complex manifold with universal covering  $\pi : B \rightarrow X$  where  $B = \{v \in \mathbb{C}^n : \|v\| < 1\}$ .*

*Let  $Z$  be a non-empty closed analytic subset of  $X$  and let  $\epsilon > 0$ . Then  $\pi^{-1}(Z)$  is not contained in any proper closed analytic subset of*

$$B_{1+\epsilon} = \{v \in \mathbb{C}^n : \|v\| < 1 + \epsilon\}.$$

*Proof.* This is immediate, because  $B_{1+\epsilon}$  is a Stein manifold containing  $B$  as relatively compact connected open subset.  $\square$

**Theorem 2.** *Let  $X$  be a reduced Stein space, and let  $D \subset X$  be a bounded domain on which a group  $G$  acts cocompactly.*

*Then  $D$  is smooth.*

*Proof.* The singular locus  $Sing(X)$  of  $X$  is a closed analytic subset of  $X$ . Since  $X$  is reduced, we have  $Sing(X) \neq X$ . Now every automorphism of  $D$  must stabilize  $Sing(D) = D \cap Sing(X)$ . Therefore the preceding corollary implies that  $Sing(D) = \{\}$ , i.e.,  $D$  is smooth.  $\square$

In particular, we obtain a negative answer to the question of Kollár discussed in the introduction:

**Theorem 3.** *Let  $X$  be a reduced Stein space with a bounded domain  $D$ . If there exists a group of automorphisms of  $D$  acting properly discontinuously on  $D$  with compact quotient, then  $D$  must be smooth.*

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